ON AUBRY SETS AND MATHER'S ACTION FUNCTIONAL

BY

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ABSTRACT

We study Lagrangian systems on a closed manifold M. We link the differentiability of Mather's β-function with the topological complexity of the complement of the Aubry set. As a consequence, when M is a closed, orientable surface, the differentiability of the β-function at a given homology class is forced by the irrationality of the homology class. This allows us to prove the two-dimensional case of a conjecture by Mañé.

1. Introduction

We start by recalling some facts about Aubry–Mather theory. Let M be a smooth, closed, connected n-dimensional manifold and L be a Lagrangian on the tangent bundle TM, that is, a $C^r, r \geq 2$ function on TM which is convex and superlinear when restricted to any fiber. The Euler–Lagrange equation then defines a flow Φ_t on TM, complete in the autonomous case. Throughout this paper we assume M to be endowed with a fixed Riemann metric, with respect to which we evaluate distances and norms in the tangent bundle; our results do not depend on the metric. Denote by π the canonical projection $TM \to M$.

For $x, y \in M$ define $h_t(x, y)$ as the minimum, over all absolutely continuous curves $\gamma \colon [0, t] \to M$ with $\gamma(0) = x$, $\gamma(t) = y$, of $\int_0^t L(\gamma, \dot{\gamma}) ds$. Then, by Fathi's weak KAM theorem ([Fa97a]) there exists $c(L) \in \mathbb{R}$ such that $\liminf_{t \to \infty} (h_t(x, y) + c(L)t)$ is finite for every x, y. This lim inf, originally defined

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in [Mr93], is called the Peierls barrier and denoted $h(x, y)$ and $c(L)$ is Mañé's critical value (see [Mn97]). The Aubry set $\mathcal{A}_0$ is then defined in [Fa97b] as the zero locus of h restricted to the diagonal in $M \times M$. The canonical projection π is a bi-Lipschitz homeomorphism between $\mathcal{A}_0$ and the set $\tilde{\mathcal{A}}_0$ of velocity vectors of orbits in $\mathcal{A}_0$ (Graph Property). Furthermore $\tilde{\mathcal{A}}_0$ is compact and Φ_t-invariant.

Fathi's weak KAM theorem asserts that there exists a Lipschitz function u_+ (resp. u_-) such that $u_\pm(\gamma(t)) - u_\pm(\gamma(0)) \leq \int_0^t (L + c(L))(\gamma, \dot{\gamma}) ds$ for every absolutely continuous path $\gamma \colon [0, t] \to M$, which is written $u \prec L + c(L)$ for short, and such that for every $x \in M$, $t \in \mathbb{R}$ there is a C^1 path $\gamma \colon [0, t] \to M$ with $\gamma(0) = x$ (resp. $\gamma(t) = x$) achieving equality. Such functions come in pairs, called conjugate pairs (u_+, u_-) such that $u_+ \leq u_-$ with equality on $\mathcal{A}_0$. Theorem 6 of [Fa97b] asserts that $h(x, y) = \sup\{u_-(y) - u_+(x)\}$, where the supremum is taken over conjugate pairs of weak KAM solutions.

For every closed 1-differential ω, $L - \omega$ is a convex and superlinear Lagrangian, we sometimes denote $\mathcal{A}_\omega$ its Aubry set $\mathcal{A}_0(L - \omega)$. Mather's α-function is defined in [Mr90] as

$$\alpha(\omega) = -\min\{\int_{TM} (L - \omega) d\mu \colon \mu \in \mathcal{M}\}$$

where $\mathcal{M}$ is the set of closed measures on TM, that is (see [Ba99]) the compactly supported probability measures μ on TM such that $\int df \, d\mu = 0$ for every C^1 function f on M. In other words, those are the measures with a well-defined homology class. The measures achieving the minimum are invariant by the Euler–Lagrange flow Φ_t of L (see [Ba99]). The quantity α defines a convex and superlinear function on $H^1(M, \mathbb{R})$, twice the squareroot of which is also called stable norm when L is a metric (see [Mt97] and the references therein). It is convex and superlinear and its Fenchel transform is Mather's β-function on $H_1(M, \mathbb{R})$, which is defined, for every real homology class h, as

$$\beta(h) = \min\{\int_{TM} (L) d\mu \colon \mu \in \mathcal{M}, [\mu] = h\}.$$

Let $\tilde{\mathcal{M}}_\omega$ be the closure in TM of the union of the supports of measures in $\mathcal{M}$ achieving the minimum in the expression of α. Such measures are called ω-minimising measures, or just minimising measures if $[\omega] = 0$. We call Mather set of L and ω, and denote $\mathcal{M}_\omega$ the projection $\pi(\tilde{\mathcal{M}}_\omega)$; it is contained in $\mathcal{A}_\omega$ ([Fa97a]). In particular we call Mather set of L the Mather set $\mathcal{M}_0$ corresponding to the zero cohomology class.

For every $[\omega] \in H^1(M, \mathbb{R})$ we call F_ω the maximal face of the epigraph Γ_α of α containing $[\omega]$ in its interior (see [Mt97]), and $\text{Vect}\, F_\omega$ the underlying vector

space of the affine subspace generated by F_ω in $H^1(M, \mathbb{R})$. Beware that $\operatorname{Vect} F_\omega$ is not, unless F_ω contains the origin, the vector space generated by F_ω. Note that $F_\omega = \{[\omega]\}$ if α is strictly convex at $[\omega]$. The value of α at the null cohomology class is Mañé's critical value $c(L)$.

In section 3 we relate the dimension of the faces of Γ_α to the topological complexity of the complement of $\mathcal{A}_\omega$ in M, as follows. Let $C_\omega(\epsilon)$ be the set of integer homology classes which are represented by a piecewise C^1 closed curve made with arcs contained in $\mathcal{A}_\omega$, except for a remainder of total length less than ϵ. Let C_ω be the intersection of $C_\omega(\epsilon)$ over all $\epsilon > 0$, and let V_ω be the vector space spanned in $H_1(M, \mathbb{R})$ by C_ω. Note that V_ω is an integer subspace of $H_1(M, \mathbb{R})$, that is, it has a basis of integer elements (images in $H_1(M, \mathbb{R})$ of elements of $H_1(M, \mathbb{Z})$).

We denote

- by $V_\omega^\perp$ the vector space of cohomology classes of one-forms of class C^1 that vanish on V_ω,
- by G_ω the vector space of cohomology classes of one-forms of class C^1 that vanish in $T_x M$ for every $x \in \mathcal{A}_\omega$,
- by E_ω the vector space of cohomology classes of one-forms of class C^1, the supports of which are disjoint from $\mathcal{A}_\omega$.

THEOREM 1: *We have $E_\omega \subset \operatorname{Vect} F_\omega \subset G_\omega \subset V_\omega^\perp$. When M is a closed, orientable surface all inclusions are equalities and, furthermore, $\operatorname{Vect} F_\omega$ is an integer subset of $H^1(M, \mathbb{R})$.*

THEOREM 2: *When M is a closed, orientable surface, the vector space $\operatorname{Vect} F_\omega$ is lower semi-continuous with respect to the Lagrangian.*

Theorem 1 means that when M is a closed, orientable surface, the dimension of the face F_ω equals the number of homologically independent closed curves disjoint from $\mathcal{A}_\omega$.

As a corollary we get differentiability results for β. The idea here was given to the author by Albert Fathi.

Let h be a homology class. A cohomology class ω is said to be a subderivative for β at h if $< \omega, h >= \beta(h) + \alpha(\omega)$. The subderivatives for β at h form a face F_h of Γ_α. By Proposition 6 the Aubry (resp. Mather) sets for all the cohomology classes in the interior of this face coincide. We call that Aubry set (resp. Mather set) the Aubry set (resp. Mather set) of h, and denote it $\mathcal{A}_h$ (resp. $\mathcal{M}_h$).

Recall that the tangent cone to the epigraph of β at h is the smallest cone in $H_1(M, \mathbb{R}) \times \mathbb{R}$ with vertex $(h, \beta(h))$ and containing the epigraph of β. We say

that the β-function is differentiable at h in the direction d if the tangent cone to the epigraph of β at h contains the affine subspace $h + \mathbb{R}d$.

Thus we say the β-function is differentiable in k directions at a homology class h if the tangent cone at h to the epigraph of β splits as a metric product of $\mathbb{R}^k$ and another cone which contains no straight line (affine subspace of dimension one).

We say a homology class h is k-irrational if k is the dimension of the smallest subspace of $H_1(M, \mathbb{R})$ generated by integer classes and containing h. In particular, 1-irrational means "on a line with rational slope" and $\dim H_1(M, \mathbb{R})$-irrational means completely irrational. We call rational any homology class of the form $1/nh$, where n is an integer and h is the image in $H_1(M, \mathbb{R})$ of an integer homology class. The integrality of Vect F_ω has the following consequence:

COROLLARY 3: *Let M be a closed, orientable surface, and L be a Lagrangian on M. At a k-irrational homology class h the β-function of L is differentiable in at least k directions.*

This was conjectured, and proved in the torus case, by V. Bangert. A similar result was proved for twist maps of the annulus by J. Mather in [Mr90]. See also [D93].

In particular, when M is a closed, orientable surface, β is differentiable in every direction at a completely irrational class. Rademacher's theorem says a convex function is differentiable almost everywhere but does not provide an explicit set of differentiability points. In [BIK97] a C^r metric is constructed on a torus of dimension $8r + 8$, such that its stable norm is not differentiable in all directions at some completely irrational class.

On the other hand, if β is differentiable in one (resp. no) direction at some homology class h, then h must be 1-irrational (resp. zero). Also note that at every non-zero class β is differentiable in the radial direction.

In the next section we investigate generic properties of Lagrangian systems. We say a property is true for a generic Lagrangian if, given a Lagrangian L, there exists a residual (countable intersection of open and dense subsets) subset $\mathcal{O}$ of $C^\infty(M)$ such that the property holds for $L + f, \forall f \in \mathcal{O}$. Mañé ([Mn96, CDI97]) proved that for a generic Lagrangian, there exists a unique minimising measure and put forth in [Mn96] the

CONJECTURE 4 (Mañé): *For a generic Lagrangian L on a closed manifold M there exists a dense open set U_0 of $H^1(M, \mathbb{R})$ such that $\forall \omega \in U_0$, $\mathcal{M}_\omega(L)$ consists of a single periodic orbit, or fixed point.*

As an application of Theorems 1, 2, and the results of [Mt97] we prove this conjecture to be true when M is a closed, orientable surface.

2. Preliminary results

Recall that by a theorem of Fathi ([Fa00], p. 104) there exists a pair of conjugate weak KAM solutions (u_+, u_-) such that u_+ and u_- coincide only on $\mathcal{A}_\omega$. The main result of this section is

PROPOSITION 5: *For every $\epsilon > 0$ there exists an integrable, non-negative function G_ϵ on M such that $G_\epsilon^{-1}(0) = \mathcal{A}_0$ and for every absolutely continuous arc $\gamma: [0, t] \longrightarrow M$ we have*

$$(1) \qquad \int_0^t (L + c(L))(\gamma, \dot{\gamma})dt \geq u_+(\gamma(t)) - u_+(\gamma(0)) + \int_0^t G_\epsilon(\gamma(t))dt - \epsilon.$$

Proof: Since M is compact and the functions h_t are equi-Lipschitz on $M \times M$ ([Mr93], see also [Fa00], p. 105), by Ascoli's theorem, for every $\epsilon > 0$ there exists $T > 0$ such that

$$\forall x, y \in M, \ t \geq T \Rightarrow h_t(x, y) \geq h(x, y) - c(L)t - \epsilon.$$

Take $T(\epsilon)$ to be the infimum of such T's.

Let $\gamma: \mathbb{R}_+ \to M$ be a C^1 arc. Take $\epsilon > 0$. Let χ_ϵ be $\epsilon / \max(1, T(\epsilon))$ times the characteristic function of the closed set $(u_- - u_+)^{-1}([2\epsilon, +\infty[)$. We prove, for all positive t,

$$(2) \qquad \int_0^t (L + c(L))(\gamma, \dot{\gamma})(s)ds \geq u_+(\gamma(t)) - u_+(\gamma(0)) + \int_0^t \chi_\epsilon(\gamma(s))ds - \epsilon.$$

The proposition follows by taking G_ϵ to be the upper bound of the functions χ_δ over all $\delta \leq \epsilon$.

Define a sequence in $\mathbb{R}_+$ by $t_0 = 0$ and

$$t_{i+1} = \max\{t \geq t_i: t - t_i \geq T(\epsilon) \text{ and } \operatorname{Leb}([t_i, t] \cap \gamma^{-1}(\operatorname{supp}(\chi_\epsilon))) \leq T(\epsilon)\}$$

where Leb denotes Lebesgue measure on $\mathbb{R}$. Observe that $\gamma(t_i) \in \operatorname{supp}(\chi_\epsilon)$, that $t_{i+1} - t_i \geq T(\epsilon)$, and that for all x between t_i and t_{i+1}

$$\int_{t_i}^x \chi_\epsilon(\gamma)(s)ds \leq \epsilon \frac{\operatorname{Leb}([t_i, x] \cap \gamma^{-1}(\operatorname{supp}(\chi_\epsilon)))}{\max(1, T(\epsilon))} \leq \epsilon.$$

We have, taking t_n to be the last t_i before t,

$$\int_0^t (L + c(L))(\gamma, \dot{\gamma})(s)ds = \sum_{t_{i+1} \leq t} \left(\int_{t_i}^{t_{i+1}} L(\gamma, \dot{\gamma})(s)ds + c(L)(t_{i+1} - t_i) \right)$$
$$+ \int_{t_n}^t L(\gamma, \dot{\gamma})(s)ds + c(L)(t - t_n),$$

thus, since $u_\pm$ are weak KAM solutions, and by the definitions of h_t, and $T(\epsilon)$,

$$\int_0^t L(\gamma, \dot{\gamma})(s)ds + c(L)t \geq \sum_{t_{i+1} \leq t} h_{t_{i+1} - t_i}(\gamma(t_i), \gamma(t_{i+1}))$$
$$+ c(L)(t_{i+1} - t_i) + u_+(\gamma(t)) - u_+(\gamma(t_n))$$
$$\geq \sum_{t_{i+1} \leq t} (h(\gamma(t_i), \gamma(t_{i+1})) - \epsilon) + u_+(\gamma(t)) - u_+(\gamma(t_n))$$
$$\geq \sum_{t_{i+1} \leq t} (u_-(\gamma(t_{i+1})) - u_+(\gamma(t_i)) - \epsilon) + u_+(\gamma(t)) - u_+(\gamma(t_n))$$
$$\geq \sum_{i=0}^n (u_-(\gamma(t_i)) - u_+(\gamma(t_i)) - \epsilon) + u_+(\gamma(t)) - u_+(\gamma(0))$$
$$\geq u_+(\gamma(t)) - u_+(\gamma(0)) + \epsilon \sharp \{i/ \ t_i \leq t\}$$
$$\geq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \chi_\epsilon(\gamma)(s))ds + u_+(\gamma(t)) - u_+(\gamma(0))$$
$$\geq u_+(\gamma(t)) - u_+(\gamma(0)) + \int_0^t \chi_\epsilon(\gamma)(s)ds - \epsilon. \qquad \blacksquare$$

2.1. PROOF OF PROPOSITION 6.   The next proposition enables us to speak of the Aubry set of a face of the epigraph of α, and therefore of the Aubry set of a homology class.

PROPOSITION 6: *If a cohomology class $[\omega_1]$ belongs to the maximal face F_ω of Γ_α containing $[\omega]$ in its interior, then $\mathcal{A}_\omega \subset \mathcal{A}_{\omega_1}$. In particular, if $[\omega_1]$ belongs to the interior of F_ω, then $\mathcal{A}_\omega = \mathcal{A}_{\omega_1}$. Conversely, if two cohomology classes ω and ω_1 are such that $\tilde{\mathcal{A}}_\omega \cap \tilde{\mathcal{A}}_{\omega_1} \neq \emptyset$, then $\alpha(\omega) = \alpha(a\omega + (1 - a)\omega_1)$ for all $a \in [0, 1]$, i.e., Γ_α has a face containing ω and ω_1.*

Proof: We can find $\omega_2 \in F_\omega$ and $a \in]0, 1[$ such that $\omega = a\omega_1 + (1 - a)\omega_2$. By [Fa98a], the following property characterises $\mathcal{A}_\omega$:

$$\forall x \in \mathcal{A}_\omega, \ \exists t_n \to +\infty, \text{ and } \gamma_n: [0, t_n] \to M, \text{ such that } \gamma(0) = \gamma(t_n) = x$$

$$\text{and } \int_0^{t_n} (L - \omega)(\gamma_n, \dot{\gamma}_n)(s)ds + \alpha(\omega)t \to 0.$$

Now $\omega = a\omega_1 + (1-a)\omega_2$, and $\alpha(\omega) = a\alpha(\omega_1) + (1-a)\alpha(\omega_2)$ since $[\omega_1], [\omega_2] \in F_\omega$. Therefore

$$a\left[\int_0^{t_n} (L - \omega_1)(\gamma_n, \dot{\gamma}_n)(s)ds + \alpha(\omega_1)t \right]$$
$$+ (1-a)\left[\int_0^{t_n} (L - \omega_2)(\gamma_n, \dot{\gamma}_n)(s)ds + \alpha(\omega_2)t \right] \to 0.$$

Observe that both summands on the left are non-negative, for if u_- is a weak KAM solution for $L - \omega_i$, $i = 1, 2$, we have

$$\int_0^{t_n} (L - \omega_i)(\gamma_n, \dot{\gamma}_n)(s)ds + \alpha(\omega_1)t \geq u_-(\gamma(t_n)) - u_-(\gamma(0))$$
$$= u_-(x) - u_-(x) = 0$$

hence $\int_0^{t_n}(L - \omega_i)(\gamma_n, \dot{\gamma}_n)(s)ds + \alpha(\omega_1)t \to 0$ when $n \to \infty$.

Conversely, let $(\gamma, \dot{\gamma})$ be an orbit in $\tilde{\mathcal{A}}_\omega \cap \tilde{\mathcal{A}}_{\omega_1}$. We have

$$\int_0^t [L - \omega + \alpha(\omega)](\gamma, \dot{\gamma})(s)ds = u(\gamma(t)) - u(\gamma(0)),$$
$$\int_0^t [L - \omega_1 + \alpha(\omega_1)](\gamma, \dot{\gamma})(s)ds = u_1(\gamma(t)) - u_1(\gamma(0)),$$

where u (resp. u_1) is a weak KAM solution for $L - \omega$ (resp. $L - \omega_1$). Therefore

$$\int_0^t [L - (a\omega + (1-a)\omega_1) + \alpha(a\omega + (1-a)\omega_1)](\gamma, \dot{\gamma})(s)ds =$$
$$a[u(\gamma(t)) - u(\gamma(0))] + (1-a)[u_1(\gamma(t)) - u_1(\gamma(0))]$$
$$+ t[\alpha(a\omega + (1-a)\omega_1) - a\alpha(\omega) - (1-a)\alpha(\omega_1))].$$

The first two summands on the right are bounded below, hence for the sum to be bounded below we must have $\alpha(\omega) = a\alpha(\omega) + (1-a)\alpha(\omega_1) = \alpha(a\omega + (1-a)\omega_1)$, since by convexity of α, $\alpha(a\omega + (1-a)\omega_1) \leq a\alpha(\omega) + (1-a)\alpha(\omega_1)$.   ∎

3. Faces of the epigraph

Proof of $G_\omega \subset V_\omega^\perp$:   It amounts to showing that a one-form in G_ω vanishes on V_ω. Let $\omega \in G_\omega$ and let h be represented as in the definition of V_ω for some $\epsilon > 0$.

Call S the part of the curve representing h which consists in segments of $\mathcal{A}_\omega$, and R the remainder. Now $< [\omega], h >= \int_S \omega + \int_R \omega$ where the first summand is zero, and the second summand can be bounded by $C\epsilon$, where C depends on L and ω only. The conclusion follows since ϵ is arbitrarily small. ∎

Proof of Vect $F_\omega \subset G_\omega$: Take $\omega_1, \omega_2 \in F_\omega$. By Proposition 6 the Aubry sets for $L - \omega_1$ and $L - \omega_2$ coincide with $\mathcal{A}_\omega$. The weak KAM solutions (u_+, u_-) are differentiable at every point of $\mathcal{A}_{(u^+, u^-)}$ (see [Fa97a]) with derivative the Legendre transform of the (well defined) tangent vector. This derivative is Lipschitz and, furthermore (see [Fa00], p. 92), we have

$$(3) \quad |u_\pm(\phi(y)) - u_\pm(\phi(x)) - \frac{\partial L - \omega}{\partial v}(\phi(x), \dot\gamma(0)) \circ D_x\phi(y - x)| \leq K |y - x|^2$$

where ϕ is a local chart on M, x and y are two points in the inverse image of $\mathcal{A}_\omega$ by the chart, $\dot\gamma(0)$ is the tangent vector to $\mathcal{A}_\omega$ at $\phi(x)$, and K only depends on the chart. So Whitney's extension theorem ([Fe69], theorem 3.1.14) allows us to take $\tilde u_1$ and $\tilde u_2$ two C^1 functions, the derivatives of which coincide with that of u_+^1 and u_+^2 respectively along $\mathcal{A}_\omega$. Replace ω_2 by $\omega_2 + d\tilde u_1 - d\tilde u_2$. This one-form coincides with ω_1 in the tangent space to every point of $\mathcal{A}_\omega$, hence the cohomology class $[\omega_1 - \omega_2]$ belongs to G_ω. ∎

Proof of $E_\omega \subset$ Vect F_ω: Assume, replacing if necessary L by $L - \omega$, that $\omega = 0$. We actually prove a slightly stronger statement. Call $\tilde{\mathcal{T}}_0$ the intersection with $\mathcal{A}_0$ of the union of Hausdorff limits, when η tends to 0, of supports of $L - \eta$-minimising measures, and call $\mathcal{T}_0$ its projection to M. Let η be supported away from $\mathcal{T}_0$.

For starters we prove that there exists $\delta > 0$ such that for all $L + \delta\eta$-minimising measure μ, for all (x, v) in $\text{supp}(\mu)$, we have $\delta|\eta_x(v)| \leq G_1(x)$ where G_1 comes from Proposition 5.

Indeed, assume otherwise. Then there exists a sequence $\delta_n \longrightarrow 0$, $L + \delta_n\eta$-minimising measures μ_n, and points (x_n, v_n) in $\text{supp}(\mu_n)$ such that for all n we have

$$(4) \qquad\qquad \delta_n|\eta_{x_n}(v_n)| > G_1(x_n).$$

The sequence (x_n, v_n) is bounded in TM because the measures μ_n sit in the energy levels $\alpha(\delta_n[\eta])$. So we may assume $(x_n, v_n) \longrightarrow (x, v)$. Then we have $G_1(x) = 0$, so $x \in \mathcal{A}_0$. Besides, (x, v) belongs to a Hausdorff limit point of the sequence of compact sets $\text{supp}(\mu_n)$, so $x \in \mathcal{T}_0$. But then for n large enough,

since η is supported outside $\mathcal{T}_0$, we should have $\eta_{x_n}(v_n) = 0$, which contradicts Equation 4 since G_1 is non-negative.

Therefore, we see that for every orbit γ in the support of an $L + \delta\eta$-minimising measure μ, by Equation 1 we have

$$\int_0^t (L + c(L) \pm \delta\eta)(\gamma, \dot{\gamma})(s)ds \geq$$

$$u_+(\gamma(t)) - u_+(\gamma(0)) + \int_0^t (G_1 \pm \delta\eta)(\gamma, \dot{\gamma})(s)ds - 1$$

so, by averaging and letting t go to infinity,

$$-c(L \pm \delta\eta) \geq \int (G_1 \pm \delta\eta)d\mu - c(L)$$

whence, since $G_1 \pm \delta\eta$ is non-negative on the support of μ,

$$\alpha(0) = c(L) \geq c(L \pm \delta\eta) = \alpha(\pm\delta\eta).$$

By convexity of α, we have $2\alpha(0) \leq \alpha(\delta\eta) + \alpha(-\delta\eta)$ so we get $\alpha(0) = \alpha(\delta\eta)$, which implies that $\delta[\eta]$ belongs to F_0. ∎

3.1. THE TWO-DIMENSIONAL CASE.   We prove that when M is a closed surface, $V_\omega^\perp \subset E_\omega$, thus proving all inclusions to be equalities. The same arguments prove that $E_{u^+,u^-} = \text{Vect } F_0$ for every conjugate pair of weak KAM solutions (u^+, u^-) for L. Since V_ω is an integer subset of $H_1(M, \mathbb{R})$ this implies that $\text{Vect } F_\omega$ is an integer subset of $H^1(M, \mathbb{R})$.

To that end we prove that there exists a neighborhood U of $\mathcal{A}_\omega$ such that every closed curve contained in U has its homology class in V_ω. First let us show how this implies the equality. If a 1-form α vanishes on every element of V_ω, then there exists a function f defined on U such that the restriction of α to U is equal to df. Extend f to M; now $\alpha - df$ is cohomologous to α and supported away from U.

Assume the surface has genus greater than one, the genus one case being treated by Bangert in [Ba94], and assume our reference metric g has negative curvature. By [BG99] every minimising orbit stays within finite distance, in the universal cover $\tilde{M}$ of M, of a g-geodesic. In particular, one can define the ends of a minimiser in the boundary at infinity of $\tilde{M}$. Call λ the geodesic lamination obtained from $\mathcal{A}_\omega$ by replacing each orbit by the corresponding geodesic.

From [CB88], we know that each boundary component of a connected component of the complementary set of λ in M is either a closed leaf of λ, or a finite

sequence of non-closed leaves $\delta_1, \ldots, \delta_n$ such that δ_i and δ_{i+1} are asymptotic (i being in $\mathbb{Z}/n\mathbb{Z}$).

Therefore, each boundary component of a connected component of the complementary set of $\mathcal{A}_\omega$ in M is either a closed orbit in $\mathcal{A}_\omega$, or a finite sequence of non-closed orbits $\delta_1, \ldots, \delta_n$ such that δ_i and δ_{i+1} are asymptotic (i being in $\mathbb{Z}/n\mathbb{Z}$).

Hence for each boundary component δ of a connected component R of the complementary set of $\mathcal{A}_\omega$ in M there exists a neighborhood V of δ in R such that every arc contained in V, with its end on δ, is homotopic, with fixed ends, to an arc consisting of portions of δ and a remainder of length arbitrarily small (or no remainder at all if δ is a closed leaf). Now we just need to take U such that $U \cap M \smallsetminus \mathcal{A}_\omega$ is contained in the union over all boundary components of R, and over all connected component of the complementary set of $\mathcal{A}_\omega$ in M, of such neighborhoods. ∎

Proof of Corollary 3:   Let h be a k-irrational homology class. Then the set of subderivatives to β at h form a face F_h of Γ_α. Furthermore, β is differentiable in $(\dim H_1(M, \mathbb{R}) - \dim F_h)$ directions. Take ω in the interior of the face F_h. We have $F_h \subset F_\omega$, so $\dim G_\omega \geq \dim F_h$.

Then for every $\omega' \in G_\omega$ we have $< \omega', h >= 0$. Note that

$$\{h \in H_1(M, \mathbb{R}): \ < \omega', h >= 0 \ \forall \omega' \in G_\omega\}$$

is an integer subset of $H_1(M, \mathbb{R})$, of dimension $\dim H_1(M, \mathbb{R}) - \dim G_\omega$.

Since h is k-irrational this implies $\dim H_1(M, \mathbb{R}) - \dim G_\omega \geq k$, whence $\dim H_1(M, \mathbb{R}) - \dim F_h \geq k$, which proves Corollary 3. ∎

Proof of Theorem 2:   Assume a sequence of Lagrangians L_n converges, in the C^2-topology, to a C^2 Lagrangian L.

Let (u_n^+, u_n^-) be a conjugate pair of weak KAM solutions for L_n. By [Fa00], p. 88 the functions (u_n^+, u_n^-) are equi-Lipschitz. By Ascoli's theorem we may assume that (u_n^+, u_n^-) converges to a pair (u^+, u^-) of Lipschitz functions. Furthermore, $u^\pm \prec L + c(L)$. Take $x \in M$ and $t \in \mathbb{R}_+$. For every $n \in \mathbb{N}$ there exists a C^1 path $\gamma_n: [0, t] \longrightarrow M$ such that $\gamma_n(0) = x$ (resp. $\gamma_n(t) = x$) and

$$u_n^\pm(\gamma_n(t)) - u_n^\pm(\gamma_n(0)) = \int_0^t (L_n + c(L_n))(\gamma_n, \dot{\gamma}_n) ds.$$

Take $v \in T_x M$ a limit point of $\dot{\gamma}_n(0)$ (resp. $\dot{\gamma}_n(t)$). Then the extremal trajectory $\gamma: [0, t] \longrightarrow M$ of the Lagrangian, with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$ (resp. $\gamma(t) = x$

and $\dot\gamma(t) = v$), is a uniform limit of γ_n and so

$$u^\pm(\gamma(t)) - u^\pm(\gamma(0)) = \int_0^t (L + c(L))(\gamma, \dot\gamma)ds.$$

This shows that (u^+, u^-) are weak KAM solutions for L. Then for every neighborhood U of $\{x \in M : u^+(x) = u^-(x)\}$ there exists an $N \in \mathbb{N}$ such that $\forall n \geq N$, $\{x \in M : u_n^+(x) = u_n^-(x)\} \subset U$. Hence there exists an $N \in \mathbb{N}$ such that $\forall n \geq N$, $E_\omega \subset E_{\omega_n}$. ∎

4. On generic Lagrangians

From now on we assume M to be a closed orientable surface. We begin with a

LEMMA 7: *Let L be a Lagrangian on a closed, orientable surface. The set $S(L)$ of subderivatives to β at 1-irrational homology classes is dense in $H^1(M, \mathbb{R})$.*

Proof: Assume there exists an open set U in $H^1(M, \mathbb{R})$ such that $U \cap S(L) = \emptyset$. We may assume U to be convex. Then the set

$$V = \{h \in H_1(M, \mathbb{R}) : \exists \omega \in U, <\omega, h> = \alpha(\omega) + \beta(h)\}$$

is also convex. Call H the vector space V generates in $H_1(M, \mathbb{R})$. Then, since V does not contain any 1-irrational class, the codimension of H is at least one. Now $U = \bigcup_{h \in V} F_h$, so there exists $h \in V$ such that $\dim F_h \geq 1$. Such an h is at most $(\dim H_1(M, \mathbb{R}) - 1)$-irrational by Corollary 3. Take ω in the interior of F_h; we have $\text{Vect}\, F_\omega = E_\omega$, so there exists a closed curve γ such that $\mathcal{A}_\omega$ is disjoint from γ. Furthermore, by semi-continuity of $\text{Vect}\, F_\omega = E_\omega$, there exists a convex neighborhood U_1 of ω in U such that for all ω' in U_1, $\mathcal{A}_0(L - \omega')$ is disjoint from γ. In particular, H is contained in the integer subspace defined by the equation $\text{Int}([\gamma], .]) = 0$.

Now assume by induction we have proved that for some

$$2 \leq k \leq \dim H_1(M, \mathbb{R}) - 2$$

there exist ω_k in U, a convex neighborhood U_k of ω_k in U, and closed curves $\gamma_1 := \gamma, \ldots, \gamma_k$ such that for all ω' in U_k, $\mathcal{A}_0(L - \omega')$ is disjoint from $\gamma_1 := \gamma, \ldots, \gamma_k$. Likewise, define V_k to be the set of homology classes at which elements of U_k are subderivatives, and H_k to be the vector space generated by V_k. Then H_k is contained in the integer subspace defined by the equations $\text{Int}([\gamma_i], .]) = 0$ for $i = 1, \ldots, k$ and the codimension of H_k is at least k. Assume the codimension of H_k is

exactly k; then as previously H_k is an integer subspace. Any open (in the induced topology) subset of such a subspace contains a 1-irrational class, an impossibility. So the codimension of H_k is at least $k + 1$. Then, as previously, there exists $h_k \in V_k$ such that $\dim F_{h_k} \geq k+1$. Such an h_k is at most $(\dim H_1(M, \mathbb{R}) - k - 1)$-irrational by Corollary 3. Take ω_{k+1} in the interior of F_{h_k}; we have $\mathrm{Vect}\, F_{\omega_{k+1}} = E_{\omega_{k+1}}$, so there exists a closed curve γ_{k+1} homologically independent from $\gamma_1 := \gamma, \dots, \gamma_k$ such that $\mathcal{A}_{\omega_{k+1}}$ is disjoint from $\gamma_1, \dots, \gamma_{k+1}$. Furthermore, by semi-continuity of $\mathrm{Vect}\, F_\omega = E_\omega$, there exists a convex neighborhood U_{k+1} of ω_{k+1} in U_k such that for all ω' in U_{k+1}, $\mathcal{A}_0(L - \omega')$ is disjoint from $\gamma_1, \dots, \gamma_{k+1}$.

By induction, we prove that U contains a $(\dim H_1(M, \mathbb{R}) - k)$-irrational class, for all $k = 1, \dots, \dim H_1(M, \mathbb{R}) - 1$, a contradiction.   ∎

By [Mt97], Proposition 5, any minimizing measure with a rational homology class must be supported on a union of periodic orbits, or fixed points.

By [Mn96], Theorem D, for a given homology class h, there exists a residual subset $\mathcal{O}_h$ of $C^\infty(M)$ such that for all $\phi \in \mathcal{O}_h$ there exists a unique closed measure in $\mathcal{M}_h(L + \phi)$.

Then for all h with rational direction, for all $\phi \in \mathcal{O}_h$ there exists a unique closed measure $\mu_{h,\phi}$ in $\mathcal{M}_h(L + \phi)$, supported on a union of periodic orbits $\gamma_{h,\phi}$. Every such periodic orbit is minimising in its homology class. Then by [Mn96], Theorem D, we may assume that $\gamma_{h,\phi}$ consists of pairwise non-homologous periodic orbits. For any given K only a finite number of integer homology classes have their L-action $\leq K$, so then $\gamma_{h,\phi}$ actually consists of a finite number of periodic orbits $\gamma_{h,\phi,i}$. For each of those orbits there exists a closed one-form ω_i such that $\gamma_{h,\phi,i}$ is the unique $L - \omega_i$-minimising measure (cf. [Mt97], Theorem 8). Then by [CI99], Theorem D, we may assume $\gamma_{h,\phi}$ to be hyperbolic in its energy level.

Next, we prove that, for all $\phi \in \mathcal{O}_h$, there exists $\epsilon(h, \phi) > 0$, such that for any $\lambda \in]1 - \epsilon(h), 1 + \epsilon(h)[$, there exists a unique closed measure $\mu_{\lambda,\phi}$ in $\mathcal{M}_{\lambda h}(L + \phi)$, supported on a union of periodic orbits $\gamma_{\lambda,\phi}$, homotopic to $\gamma_{h,\phi}$.

Indeed, fix $\phi \in \mathcal{O}_h$, and consider a sequence λ_n of real numbers converging to one. Let μ_n be $\lambda_n h$-minimising measures. The sequence of measures μ_n converges to an h-minimising measure, and the only possibility is that it is supported on $\gamma_{h,\phi}$. The latter being hyperbolic, a topological conjugacy argument proves our claim.

The set of 1-irrational homology classes is a countable union of lines. Choose a countable dense subset h_i, $i \in \mathbb{N}$. Call $\mathcal{O}$ the intersection over all $i \in \mathbb{N}$ of $\mathcal{O}_{h_i}$; this is a countable intersection of residual sets, hence residual. Now for all $\phi \in \mathcal{O}$, there exists an open and dense subset $U(\phi)$ of the subset of 1-irrational homology

classes, such that for any $h \in U(\phi)$, there exists a unique closed measure $\mu_{h,\phi}$ in $\mathcal{M}_h(L + \phi)$, supported on a union of periodic orbits $\gamma_{h,\phi}$.

If M has genus ≥ 2, by Theorems 7 and 8 of [Mt97], every subderivative to β at a 1-irrational homology class is contained in a face of codimension one, whether on the boundary or in the interior. By Corollary 3, if a cohomology class is contained in a face of codimension one (resp. zero), then it must be subderivative to β at a 1-irrational (resp. zero) homology class.

The same is true if M is a torus and $\phi \in \mathcal{O}$; for in that case, in every 1-irrational homology class h, there exists a unique minimising measure. Such a measure is supported on one periodic orbit, hence β is not differentiable at h ([Ba94]).

Hence when $\phi \in \mathcal{O}$, $S(L + \phi)$ equals the set of cohomology class contained in a face of codimension one or zero.

Now consider the set $S'(L + \phi)$ of cohomology classes contained in the interior of a face of codimension one or zero, and subderivative to β at a point of $U(\phi)$. By Theorem 2, $S'(L + \phi)$ is open in $H^1(M, \mathbb{R})$ for any Lagrangian L. Besides, since the interior of any face is dense in that face, and the h_i are dense in $H_1(m, \mathbb{R})$, $S'(L)$ is dense in $S(L)$, hence in $H^1(M, \mathbb{R})$. Note that for all $\omega \in S'(L + \phi)$, $\mathcal{M}_\omega$ consists of periodic orbits with the same homology class, or fixed points. Indeed if $\mathcal{M}_\omega$ contained two homologically distinct periodic orbits, then V_ω would contain their homology classes and its dimension would be at least two, so ω could not lie in the the interior of a face of codimension one or zero.

In particular, for all $\phi \in \mathcal{O}$, $\omega \in S'(L + \phi)$, $\mathcal{M}_\omega(L + \phi)$ consists of one periodic orbit or fixed point. This proves Conjecture 4 for surfaces.

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